COLOURED EXTENSION OF $GL_q(2)$ AND ITS DUAL ALGEBRA

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We address the problem of duality between the coloured extension of the quantised algebra of functions on a group and that of its quantised universal enveloping algebra *i.e.* its dual. In particular, we derive explicitly the algebra dual to the coloured extension of $GL_q(2)$ using the coloured RLL relations and exhibit its Hopf structure. This leads to a coloured generalisation of the R-matrix procedure to construct a bicovariant differential calculus on the coloured version of $GL_q(2)$. In addition, we also propose a coloured generalisation of the geometric approach to quantum group duality given by Sudbery and Dobrev.

1. INTRODUCTION

The quantum group $GL_q(2)$ is known to admit a coloured extension by introducing some continuously varying colour parameters associated to the generators. In such an extension, the associated algebra and the coalgebra are defined in a way that all Hopf algebraic properties remain preserved. Such extensions have been introduced in [1, 2, 3] and studied by various authors [4, 5, 6, 7] in recent years. However, some of the basic algebro-geometric structure underlying these coloured extensions still needs to be established. As such, we shall

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focus on the coloured extension of the most intuitive quantum group $GL_q(2)$. While some aspects of this example have already been studied from both, the standard q-deformations as well as the Jordanian (nonstandard) h-deformations [6, 7], it has only recently been shown [8] that the contraction procedure could be used to obtain the coloured Jordanian quantum groups from their coloured q-deformed counterparts. In particular, the coloured extension of $GL_q(2)$ was treated in detail in [8] to obtain a new coloured extension of Jordanian $GL_h(2)$. In the present paper, we investigate the algebra dual to the coloured extension of $GL_q(2)$ by generalising two well-known approaches to the problem: the (algebraic) R-matrix approach [9] and the geometric approach [10, 11] due to Sudbery and Dobrev. We first clarify the notion of duality between a coloured quantum group and its dual *i.e.* the coloured quantised universal enveloping algebra. We then generalise the R-matrix approach to establish duality for the coloured extension of $GL_q(2)$ and we obtain a new coloured quantum algebra corresponding to $gl(2)$ and exhibit its Hopf algebra structure. The coloured R-matrix procedure naturally leads us to formulate a constructive differential calculus [12] on the coloured extension of $GL_q(2)$.

Furthermore, we propose a coloured generalisation of the geometric notion of duality for quantum groups *i.e.* regarding the dual algebra as the algebra of tangent vectors at the identity of the group. This generalisation could also be of significance in establishing the duality for the coloured extension of Jordanian quantum groups.

2. COLOURED EXTENSION OF $GL_Q(2)$

The coloured extension of the quantum group $GL_q(2)$ is governed by the coloured R-matrix [4],

$$R_q^{\lambda,\mu} = \begin{pmatrix} q^{1-(\lambda-\mu)} & 0 & 0 & 0 \\ 0 & q^{\lambda+\mu} & 0 & 0 \\ 0 & q-q^{-1} & q^{-(\lambda+\mu)} & 0 \\ 0 & 0 & 0 & q^{1+(\lambda-\mu)} \end{pmatrix} \tag{1}$$

which is nonadditive *i.e.* $R^{\lambda,\mu} \neq R(\lambda - \mu)$. It satisfies the so-called coloured quantum Yang-Baxter equation

$$R_{12}^{\lambda,\mu} R_{13}^{\lambda,\nu} R_{23}^{\mu,\nu} = R_{23}^{\mu,\nu} R_{13}^{\lambda,\nu} R_{12}^{\lambda,\mu} \tag{2}$$

which is in general multicomponent and λ, μ, ν are considered as 'colour' variables. The RTT relations are also extended to incorporate the coloured extension as

$$R_q^{\lambda,\mu} T_{1\lambda} T_{2\mu} = T_{2\mu} T_{1\lambda} R_q^{\lambda,\mu} \tag{3}$$

(where $T_{1\lambda} = T_\lambda \otimes \mathbf{1}$ and $T_{2\mu} = \mathbf{1} \otimes T_\mu$) in which the entries of the T matrices carry colour dependence *i.e.* $T_\lambda = \left(\begin{smallmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{smallmatrix}\right)$, $T_\mu = \left(\begin{smallmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{smallmatrix}\right)$. The coproduct and counit for the coalgebra structure are given by $\Delta(T_\lambda) = T_\lambda \dot{\otimes} T_\lambda$, $\varepsilon(T_\lambda) = \mathbf{1}$. The quantum determinant $D_\lambda = a_\lambda d_\lambda - r^{-(1+2\lambda)} c_\lambda b_\lambda$ is group-like but not central. The antipode is given by

$$S(T_\lambda) = D_\lambda^{-1} \begin{pmatrix} d_\lambda & -r^{1+2\lambda} b_\lambda \\ -r^{-1-2\lambda} c_\lambda & a_\lambda \end{pmatrix} \tag{4}$$

and depends on one colour variable at a time. The full Hopf algebraic structure can be constructed resulting in a coloured extension of $GL_q(2)$ within the framework of the FRT formalism. Since λ and μ are continuous variables, this implies the coloured extension of $GL_q(2)$ has an infinite number of generators. The colourless limit $\lambda = \mu = 0$ gives back the ordinary single-parameter deformed quantum group $GL_q(2)$, and the monochromatic limit $\lambda = \mu \neq 0$ gives rise to the uncoloured two-parameter deformed quantum group $GL_{p,q}(2)$.

3. DUALITY (R-MATRIX APPROACH)

In this section, we investigate in detail the dual structure for the coloured extension of $GL_q(2)$ employing the R-matrix approach. In doing so, let us denote the generators of the yet unknown dual algebra by $\{A_\lambda, B_\lambda, C_\lambda, D_\lambda\}$ and $\{A_\mu, B_\mu, C_\mu, D_\mu\}$. The following pairings hold

$$\langle A_{\lambda|\mu}, a_{\lambda|\mu} \rangle = \langle B_{\lambda|\mu}, b_{\lambda|\mu} \rangle = \langle C_{\lambda|\mu}, c_{\lambda|\mu} \rangle = \langle D_{\lambda|\mu}, d_{\lambda|\mu} \rangle = \mathbf{1} \tag{5}$$

All other pairings give zeroes and the notation $\lambda|\mu$ in the subscript in the above relations means *either* λ *or* μ. The R^+ and R^- matrices corresponding to the coloured extension of $GL_q(2)$ are

$$R^+ = c^+ q^{1/2} \begin{pmatrix} q^{-1/2}q^{1-\lambda+\mu} & 0 & 0 & 0 \\ 0 & q^{-1/2}q^{-(\lambda+\mu)} & q^{-1/2}(q-q^{-1}) & 0 \\ 0 & 0 & q^{-1/2}q^{\lambda+\mu} & 0 \\ 0 & 0 & 0 & q^{-1/2}q^{1+\lambda-\mu} \end{pmatrix} \tag{6}$$

$$R^- = c^- q^{-1/2} \begin{pmatrix} q^{1/2}q^{-(1-\lambda+\mu)} & 0 & 0 & 0 \\ 0 & q^{1/2}q^{-(\lambda+\mu)} & 0 & 0 \\ 0 & -q^{1/2}(q-q^{-1}) & q^{1/2}q^{\lambda+\mu} & 0 \\ 0 & 0 & 0 & q^{1/2}q^{-(1+\lambda-\mu)} \end{pmatrix} \tag{7}$$

where $R^+ = c^+ R_{21}$ and $R^- = c^- R_{12}^{-1}$ by definition. The coloured $L^\pm$ functionals can be expressed as

$$L^+_{\lambda(\mu)} = c^+ q^{1/2} \begin{pmatrix} q^{H_{\lambda(\mu)}/2} q^{\mu H_{\lambda(\mu)} - \lambda H'_{\lambda(\mu)}} & q^{-1/2}(q-q^{-1})C_{\lambda(\mu)} \\ 0 & q^{-H_{\lambda(\mu)}/2} q^{\mu H_{\lambda(\mu)} + \lambda H'_{\lambda(\mu)}} \end{pmatrix} \tag{8}$$

$$L^-_{\lambda(\mu)} = c^- q^{-1/2} \begin{pmatrix} q^{-H_{\lambda(\mu)}/2} q^{\lambda H_{\lambda(\mu)} - \mu H'_{\lambda(\mu)}} & 0 \\ q^{1/2}(q^{-1}-q)B_{\lambda(\mu)} & q^{H_{\lambda(\mu)}/2} q^{\lambda H_{\lambda(\mu)} + \mu H'_{\lambda(\mu)}} \end{pmatrix} \tag{9}$$

where $H_\lambda = A_\lambda - D_\lambda$, $H'_\lambda = A_\lambda + D_\lambda$ and $H_\mu = A_\mu - D_\mu$, $H'_\mu = A_\mu + D_\mu$. The notation $\lambda(\mu)$ in the subscript means λ (*respectively* μ). So, $L^+_{\lambda(\mu)}$ means L^+_λ (*resp.* L^+_μ) Each one of $L^\pm_\lambda$ and $L^\pm_\mu$ depends on both λ and μ. The notation $L^\pm_\lambda$ implies that the generators of the dual carry λ dependence, and similarly $L^\pm_\mu$ implies that the generators of the dual carry μ dependence. The duality pairings are then given by the action of the functionals $L^\pm_\lambda$ and $L^\pm_\mu$ on the T-matrices T_λ and T_μ

$$(L^+_{\lambda|\mu})^a_b (T_{\lambda|\mu})^c_d = (R^+)^{ac}_{bd} \tag{10}$$

$$(L^-_{\lambda|\mu})^a_b (T_{\lambda|\mu})^c_d = (R^-)^{ac}_{bd} \tag{11}$$

Again, according to the notation introduced $T_{\lambda|\mu}$ implies T_λ *or* T_μ and $L^\pm_{\lambda|\mu}$ implies $L^\pm_\lambda$ *or* $L^\pm_\mu$. For vanishing colour variables, the coloured $L^\pm$ functionals reduce to the ordinary

$L^{\pm}$ functionals for $GL_q(2)$. The commutation relations of the algebra dual to a coloured quantum group can be obtained from the modified or the *coloured* RLL relations

$$R_{12}L^{\pm}_{2\lambda}L^{\pm}_{1\mu} = L^{\pm}_{1\mu}L^{\pm}_{2\lambda}R_{12} \tag{12}$$

$$R_{12}L^{+}_{2\lambda}L^{-}_{1\mu} = L^{-}_{1\mu}L^{+}_{2\lambda}R_{12} \tag{13}$$

using the coloured $L^{\pm}$ functionals where $L^{\pm}_{1\mu} = L^{\pm}_{\mu} \otimes \mathbf{1}$ and $L^{\pm}_{2\lambda} = \mathbf{1} \otimes L^{\pm}_{\lambda}$. Using the above formulae, we obtain the commutation relations between the generating elements of the algebra dual to the coloured extension of $GL_q(2)$.

$$\begin{array}{lll} [A_\lambda, B_\mu] = B_\mu & [D_\lambda, B_\mu] = -B_\mu & \\ [A_\lambda, C_\mu] = -C_\mu & [D_\lambda, C_\mu] = C_\mu & \\ [A_\lambda, D_\mu] = 0 & [H_\lambda, H_\mu] = 0 & [H'_\lambda, \bullet] = 0 \end{array} \tag{14}$$

$$q^{-(\lambda+\mu)}C_\lambda B_\mu - q^{\lambda+\mu}B_\mu C_\lambda = \frac{q^{\lambda H_\mu + \mu H_\lambda}}{q - q^{-1}} \left[q^{-\frac{1}{2}(H_\lambda + H_\mu)} q^{\lambda H'_\lambda - \mu H'_\mu} - q^{\frac{1}{2}(H_\lambda + H_\mu)} q^{-\lambda H'_\lambda + \mu H'_\mu} \right] \tag{15}$$

$$\begin{aligned} A_\lambda A_\mu &= A_\mu A_\lambda \\ B_\lambda B_\mu &= q^{2(\mu-\lambda)} B_\mu B_\lambda \\ C_\lambda C_\mu &= q^{2(\lambda-\mu)} C_\mu C_\lambda \\ D_\lambda D_\mu &= D_\mu D_\lambda \end{aligned} \tag{16}$$

where H_λ and H'_λ are as before. The relations satisfy the $\lambda \leftrightarrow \mu$ exchange symmetry. The associated coproduct of the elements of the dual algebra is given by

$$\Delta(A_{\lambda(\mu)}) = A_{\lambda(\mu)} \otimes \mathbf{1} + \mathbf{1} \otimes A_{\lambda(\mu)} \tag{17}$$

$$\Delta(B_{\lambda(\mu)}) = B_{\lambda(\mu)} \otimes q^{A_{\lambda(\mu)} - D_{\lambda(\mu)}} + \mathbf{1} \otimes B_{\lambda(\mu)} \tag{18}$$

$$\Delta(C_{\lambda(\mu)}) = C_{\lambda(\mu)} \otimes q^{A_{\lambda(\mu)} - D_{\lambda(\mu)}} + \mathbf{1} \otimes C_{\lambda(\mu)} \tag{19}$$

$$\Delta(D_{\lambda(\mu)}) = D_{\lambda(\mu)} \otimes \mathbf{1} + \mathbf{1} \otimes D_{\lambda(\mu)} \tag{20}$$

The counit $\varepsilon(Y_{\lambda|\mu}) = 0$; where $Y_{\lambda(\mu)} = \{A_{\lambda(\mu)}, B_{\lambda(\mu)}, C_{\lambda(\mu)}, D_{\lambda(\mu)}\}$ and the antipode is

$$S(A_{\lambda(\mu)}) = -A_{\lambda(\mu)} \tag{21}$$

$$S(B_{\lambda(\mu)}) = -B_{\lambda(\mu)} q^{-(A_{\lambda(\mu)} - D_{\lambda(\mu)})} \tag{22}$$

$$S(C_{\lambda(\mu)}) = -C_{\lambda(\mu)} q^{-(A_{\lambda(\mu)} - D_{\lambda(\mu)})} \tag{23}$$

$$S(D_{\lambda(\mu)}) = -D_{\lambda(\mu)} \tag{24}$$

Thus we have defined a new single-parameter coloured quantum algebra corresponding to $gl(2)$, which in the monochromatic limit defines the standard uncoloured two-parameter quantum algebra for $gl(2)$.

4. CONSTRUCTIVE CALCULUS

We now proceed towards a coloured generalisation of the constructive differential calculus [12, 13] for the coloured extension of $GL_q(2)$. Analogous to the standard uncoloured quantum group, a bimodule Γ (space of quantum one-forms ω) is characterised by the commutation relations between ω and $a_{\lambda(\mu)} \in \mathcal{A}$, the *coloured* quantum group corresponding to $GL_q(2)$

$$\omega a_{\lambda(\mu)} = (\mathbf{1} \otimes f_{\lambda,\mu})\Delta(a_{\lambda(\mu)})\omega \tag{25}$$

and the linear functional $f_{\lambda,\mu}$ is defined in terms of the coloured $L^{\pm}$ matrices

$$f_{\lambda,\mu} = S(L^{+}_{\lambda|\mu})L^{-}_{\lambda|\mu} \tag{26}$$

Thus, we have

$$\omega a_{\lambda(\mu)} = [(\mathbf{1} \otimes S(L^{+}_{\lambda|\mu})L^{-}_{\lambda|\mu})\Delta(a_{\lambda(\mu)})]\omega \tag{27}$$

In terms of components, this can be written as

$$\omega_{ij} a_{\lambda(\mu)} = [(\mathbf{1} \otimes S(l^{+}_{(\lambda|\mu)ki})l^{-}_{(\lambda|\mu)jl})\Delta(a_{\lambda(\mu)})]\omega_{kl} \tag{28}$$

using the expressions $L^{\pm} = l^{\pm}_{ij}$ and $\omega = \omega_{ij}$ where $i, j = 1, 2$. From these relations, one can obtain the commutation relations of all the left-invariant one-forms with the elements of the coloured extension of $GL_q(2)$ The left-invariant vector fields χ_{ij} on $\mathcal{A}$ are given by the expression

$$\chi_{ij} = S(l^{+}_{(\lambda|\mu)ik})l^{-}_{(\lambda|\mu)kj} - \delta_{ij}\varepsilon \tag{29}$$

The vector fields act on the elements $a_{\lambda(\mu)}$ of the coloured quantum group as

$$\chi_{ij} a_{\lambda(\mu)} = (S(l^{+}_{(\lambda|\mu)ik})l^{-}_{(\lambda|\mu)kj} - \delta_{ij}\varepsilon)a_{\lambda(\mu)} \tag{30}$$

Furthermore, using the formula $\mathbf{d}a_{\lambda(\mu)} = \sum_i(\chi_i * a_{\lambda(\mu)})\omega^i$, we obtain the action of the exterior derivative on the generating elements

$$\begin{aligned}\mathbf{d}a_{\lambda(\mu)} &= (\mathsf{s}q^{-2+2(\lambda-\mu)} - 1)a_{\lambda(\mu)}\omega^1 + \mathsf{s}(q^{-1} - q)q^{\lambda+\mu}b_{\lambda(\mu)}\omega^+ \\ &\quad +(\mathsf{s} - 1)a_{\lambda(\mu)}\omega^2\end{aligned} \tag{31}$$

$$\begin{aligned}\mathbf{d}b_{\lambda(\mu)} &= (\mathsf{s}(q^{-1} - q)^2 + \mathsf{s} - 1)b_{\lambda(\mu)}\omega^1 + \mathsf{s}(q^{-1} - q)q^{-(\lambda+\mu)}a_{\lambda(\mu)}\omega^- \\ &\quad +(\mathsf{s}q^{-2+2(\mu-\lambda)} - 1)b_{\lambda(\mu)}\omega^2\end{aligned} \tag{32}$$

$$\begin{aligned}\mathbf{d}c_{\lambda(\mu)} &= (\mathsf{s}q^{-2+2(\lambda-\mu)} - 1)c_{\lambda(\mu)}\omega^1 + \mathsf{s}(q^{-1} - q)q^{\lambda+\mu}d_{\lambda(\mu)}\omega^+ \\ &\quad +(\mathsf{s} - 1)c_{\lambda(\mu)}\omega^2\end{aligned} \tag{33}$$

$$\begin{aligned}\mathbf{d}d_{\lambda(\mu)} &= (\mathsf{s}(q^{-1} - q)^2 + \mathsf{s} - 1)d_{\lambda(\mu)}\omega^1 + \mathsf{s}(q^{-1} - q)q^{-(\lambda+\mu)}c_{\lambda(\mu)}\omega^- \\ &\quad +(\mathsf{s}q^{-2+2(\mu-\lambda)} - 1)d_{\lambda(\mu)}\omega^2\end{aligned} \tag{34}$$

where $\omega^1 = \omega_{11}$, $\omega^+ = \omega_{12}$, $\omega^- = \omega_{21}$, $\omega^2 = \omega_{22}$ and $\mathsf{s} = (c^+)^{-1}c^-$. $\mathbf{d}\mathcal{A}$ generates Γ as a left $\mathcal{A}$-module. This defines a first order differential calculus $(\Gamma, \mathbf{d})$ on the coloured extension of $GL_q(2)$. Since the colour variables λ and μ are continuously varying, the differential calculus obtained is infinite-dimensional. The differential calculus on the uncoloured single-parameter quantum group $GL_q(2)$ is easily recovered in the colourless limit and that of the uncoloured two-parameter quantum group $GL_{p,q}(2)$ in the monochromatic limit.

5. DUAL BASIS (GEOMETRIC APPROACH)

It is well-known that two bialgebras $\mathcal{U}$ and $\mathcal{A}$ are in duality if there exists a doubly nondegenerate bilinear form

$$\langle , \rangle : \mathcal{U} \otimes \mathcal{A} \to \mathbf{C}; \quad \langle , \rangle : (u, a) \to \langle u, a\rangle; \quad \forall u \in \mathcal{U}, a \in \mathcal{A} \tag{35}$$

such that for $u, v \in \mathcal{U}$ and $a, b \in \mathcal{A}$, we have

$$\begin{aligned}\langle u, ab\rangle &= \langle \Delta_{\mathcal{U}}(u), a \otimes b\rangle \\ \langle uv, a\rangle &= \langle u \otimes v, \Delta_{\mathcal{A}}(a)\rangle\end{aligned} \tag{36}$$

$$\begin{aligned}\langle \mathbf{1}_{\mathcal{U}}, a\rangle &= \varepsilon_{\mathcal{A}}(a)\\ \langle u, \mathbf{1}_{\mathcal{A}}\rangle &= \varepsilon_{\mathcal{U}}(u)\end{aligned} \tag{37}$$

For the two bialgebras to be in duality as Hopf algebras, $\mathcal{U}$ and $\mathcal{A}$ further satisfy

$$\langle S_{\mathcal{U}}(u), a\rangle = \langle u, S_{\mathcal{A}}(a)\rangle \tag{38}$$

It is enough to define the pairing between the generating elements of the two algebras. Pairing for any other elements of $\mathcal{U}$ and $\mathcal{A}$ follows from these relations and the bilinear form inherited by the tensor product. The geometric approach for duality for quantum groups was motivated by the fact that, at the classical level, an element of the Lie algebra corresponding to a Lie group is a tangent vector at the identity of the Lie group. Let $\mathcal{H}$ be a given Hopf algebra generated by non-commuting elements a, b, c, d. The q-analogue of tangent vector at the identity would then be obtained by first differentiating the elements of the given Hopf algebra $\mathcal{H}$ (polynomials in a, b, c, d) and then putting $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1\end{smallmatrix}\right)$ later on (i.e., taking the counit operation analogous to the unit element at the group level). The elements thus obtained would belong to the dual Hopf algebra $\mathcal{H}^*$. The approach is due to Sudbery [10] and Dobrev [11] and has proved to be quite a powerful tool in understanding the quantum group duality from a geometric point of view. In what follows in this section, we propose to give a coloured generalisation of such a geometric picture of duality using the example of $GL_q(2)$. Let $\mathcal{A}_q^{\lambda,\mu}$ denote the coloured extension of $GL_q(2)$. Then, as a Hopf algebra $\mathcal{A}_q^{\lambda,\mu}$ is generated by elements $y_\lambda = \{a_\lambda, b_\lambda, c_\lambda, d_\lambda\}$ and $y_\mu = \{a_\mu, b_\mu, c_\mu, d_\mu\}$. The dual basis is given by all monomials of the form

$$\begin{aligned} g_\lambda = g_{\lambda;klmn} &= a_\lambda^k d_\lambda^l b_\lambda^m c_\lambda^n\\ g_\mu = g_{\mu;klmn} &= a_\mu^k d_\mu^l b_\mu^m c_\mu^n\end{aligned} \tag{39}$$

where $k, l, m, n \in Z_+$, and δ_{0000} is the unit of the algebra $\mathbf{1}_{\mathcal{A}}$. We use a normal ordering as follows; first put the diagonal elements from the $T_{\lambda(\mu)}$- matrix then use the lexicographic order for the others. Let $\mathcal{U}_q^{\lambda,\mu}$ be the algebra generated by tangent vectors at the identity of $\mathcal{A}_q^{\lambda,\mu}$. Then $\mathcal{U}_q^{\lambda,\mu}$ is dually paired with $\mathcal{A}_q^{\lambda,\mu}$. The pairing is defined through the coloured

q-tangent vectors as follows

$$\langle Y_\lambda, g_\lambda\rangle = \frac{\partial g_\lambda}{\partial y_\lambda}|_{\left(\begin{smallmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{smallmatrix}\right)=\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \varepsilon(\frac{\partial g_\lambda}{\partial y_\lambda}) \tag{40}$$

$$\langle Y_\mu, g_\lambda\rangle = \frac{\partial g_\lambda}{\partial y_\lambda}|_{\left(\begin{smallmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{smallmatrix}\right)=\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \varepsilon(\frac{\partial g_\lambda}{\partial y_\lambda}) \tag{41}$$

$$\langle Y_\lambda, g_\mu\rangle = \frac{\partial g_\mu}{\partial y_\mu}|_{\left(\begin{smallmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{smallmatrix}\right)=\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \varepsilon(\frac{\partial g_\mu}{\partial y_\mu}) \tag{42}$$

$$\langle Y_\mu, g_\mu\rangle = \frac{\partial g_\mu}{\partial y_\mu}|_{\left(\begin{smallmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{smallmatrix}\right)=\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} = \varepsilon(\frac{\partial g_\mu}{\partial y_\mu}) \tag{43}$$

where $Y_\lambda = \{A_\lambda, B_\lambda, C_\lambda, D_\lambda\}$ and $Y_\mu = \{A_\mu, B_\mu, C_\mu, D_\mu\}$ are the sets of generating elements of the dual algebra (which has unit $\mathbf{1}_\mathcal{U}$). More compactly, one can write

$$\langle Y_{\lambda|\mu}, g_{\lambda(\mu)}\rangle = \varepsilon(\frac{\partial g_{\lambda(\mu)}}{\partial y_{\lambda(\mu)}}) \tag{44}$$

Explicitly, we obtain

$$\langle A_{\lambda|\mu}, g_{\lambda(\mu)}\rangle = \varepsilon(\frac{\partial g_{\lambda(\mu)}}{\partial a_{\lambda(\mu)}}) = k\delta_{m0}\delta_{n0} \tag{45}$$

$$\langle B_{\lambda|\mu}, g_{\lambda(\mu)}\rangle = \varepsilon(\frac{\partial g_{\lambda(\mu)}}{\partial b_{\lambda(\mu)}}) = \delta_{m1}\delta_{n0} \tag{46}$$

$$\langle C_{\lambda|\mu}, g_{\lambda(\mu)}\rangle = \varepsilon(\frac{\partial g_{\lambda(\mu)}}{\partial c_{\lambda(\mu)}}) = \delta_{m0}\delta_{n1} \tag{47}$$

$$\langle D_{\lambda|\mu}, g_{\lambda(\mu)}\rangle = \varepsilon(\frac{\partial g_{\lambda(\mu)}}{\partial d_{\lambda(\mu)}}) = l\delta_{m0}\delta_{n0} \tag{48}$$

where differentiation is from the right. As a consequence of the above pairings, the following relations hold

$$\langle A_{\lambda|\mu}, T_{\lambda|\mu}\rangle = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right) \tag{49}$$

$$\langle B_{\lambda|\mu}, T_{\lambda|\mu}\rangle = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \tag{50}$$

$$\langle C_{\lambda|\mu}, T_{\lambda|\mu}\rangle = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right) \tag{51}$$

$$\langle D_{\lambda|\mu}, T_{\lambda|\mu}\rangle = \left(\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right) \tag{52}$$

where $T_\lambda = \left(\begin{smallmatrix} a_\lambda & b_\lambda \\ c_\lambda & d_\lambda \end{smallmatrix}\right)$ and $T_\mu = \left(\begin{smallmatrix} a_\mu & b_\mu \\ c_\mu & d_\mu \end{smallmatrix}\right)$ as before. Furthermore,

$$\langle Y_{\lambda|\mu}, \mathbf{1}_\mathcal{A}\rangle = 0; \quad \langle \mathbf{1}_\mathcal{U}, g_{\lambda|\mu}\rangle = \varepsilon_\mathcal{A}(g_{\lambda|\mu}) = \delta_{m0}\delta_{n0} \tag{53}$$

The action of the monomials in $\mathcal{U}_q^{\lambda,\mu}$ on g_λ and g_μ then lead to the coloured q-commutation relations between the generators of the dual algebra.

6. CONCLUDING REMARKS

We have investigated the structure of the coloured extension of the quantum group $GL_q(2)$ and its dual algebra. After establishing the notion of duality, the dual algebra has been derived explicitly using the R-matrix approach. We not only obtain a new coloured quantum algebra corresponding to $gl(2)$ but also show that such a coloured generalisation of the R-matrix approach leads to formulating a constructive differential calculus for the coloured case. The colourless and the monochromatic limits of both the dual algebra as well as the differential calculus are in agreement with already known results for $GL_q(2)$ and $GL_{p,q}(2)$. In the later part of the paper, we have proposed a generalisation of the geometric picture of duality to incorporate the coloured extensions of quantum groups. It would be interesting to investigate in detail this setting in the context of the coloured Jordanian quantum groups. The results easily extend to the higher-dimensional and multi-parametric cases.

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